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Sufficiency in Non Linear Programming: A Unified Problem

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Abstract: **Sufficient optimality conditions are derived for nonlinear programming problems in presence of both equality and inequality constraints. This result subsumes the well-known Kuhn-Tucker and Fritz John sufficiency theorems discussed separately in the literature by several authors.**

Keywords: **Nonlinear programming problems.**

1. INTRODUCTION

Consider the following mathematical programming problem:

(MP) Minimize $f(x)$,

subject to $x \in X = \{x \in X^{\circ} : g(x) \leqq 0, h(x) = 0\}$,

where X° is an open subset of R^n and

$$f : X^{\circ} \to R, \; g : X^{\circ} \to R^m, \; h : X^{\circ} \to R^k$$

are differentiable at $\bar{x} \in X$.

Sufficient optimality conditions have played an important role in theoretical as well as computational developments of mathematical programming and have been studied extensively in the literature. Mangasarian [4] assuming f to be pseudoconvex, g_I (where $I = \{i : g_i(x) = 0\}$) to be quasiconvex and h to be both quasiconvex and quasiconcave at $x \varepsilon X^0$ showed that the Kuhn-Tucker type sufficient conditions for $x \varepsilon X^0$ to be optimal for MP are the existence of $u \varepsilon R^m$ and $v \varepsilon R^k$ Satisfying

$$\nabla f(\bar{x}) + \bar{u} \nabla g(\bar{x}) + \bar{v} \nabla h(\bar{x}) = 0,$$
$$\bar{u} g(\bar{x}) = 0,$$
$$g(\bar{x}) \leqq 0,$$
$$h(\bar{x}) = 0,$$
$$\bar{u} \geqq 0,$$

Bhatt and Misra [3] proved that if, f, g and h are convex at $x \varepsilon X^0$ then the above conditions with the additional restriction $v \geq 0$ are sufficient for x to be optimal for Mathematical Programming. Bector and Gulati [2] and Singh [5] obtained this result assuming pseudo-convexity of f and quasiconvexity of g and h at $x \varepsilon X^0$.

Bhatt and Misra also showed that under convexity of *f* and strict convexity of *g* and *h*, the Fritz John type sufficient optimality conditions for Mathematical Programming are the existence of $u_0 \varepsilon R$, $u \varepsilon R^m$ and $v \varepsilon R^k$ satisfying

$$\bar{u}_0 \nabla f(\bar{x}) + \bar{u} \nabla g(\bar{x}) + \bar{v} \nabla h(\bar{x}) = 0,$$
$$\bar{u} g(\bar{x}) = 0,$$
$$g(\bar{x}) \leqq 0,$$
$$h(\bar{x}) = 0,$$
$$(\bar{u}_0, \bar{u}, \bar{v}) \geqslant 0.$$

Bector and Gulati [2] and Singh [5] established this result under generalized convexity assumptions. The modified Fritz John optimality criterion in Skarpness and Sposito was also presented in [2].

Several other authors have also discussed Kuhn-Tucker and Fritz John type sufficient optimality conditions separately. The aim of the present note is to present unified sufficient optimality criteria which subsumes Kuhn-Tucker and Fritz John optimality criteria. Moreover, unlike the proof of Fritz John sufficient optimality theorems [2-6], the proof presented here does not require any theorem of alternative and therefore is simple.

2. UNIFIED SUFFICIENT OPTIMALITY CRITERION

We first state the following definition [1, 6]:

Definition. A numerical function *f*, defined on an open set X^0 in R^n, which is differentiable at x, is said to be strictly pseudoconvex at x if, for each x ε Xo, x $\equiv$ x.

$$\nabla f(\bar{x})\,(x - \bar{x}) \geqq 0 \Rightarrow f(x) > f(\bar{x}),$$

or equivalently

$$f(x) \leqq f(\bar{x}) \Rightarrow \nabla f(\bar{x})\,(x - x^{\circ}) < 0.$$

THEOREM. *Let $\bar{x} \in X^{\circ}$ and f be pseudoconvex, g_I and h be quasiconvex at $\bar{x}$. If there exist $\bar{u}_0 \in R$, $\bar{u} \in R^m$, $\bar{v} \in R^k$ satisfying*

$$\bar{u}_0 \nabla f(\bar{x}) + \bar{u} \nabla g(\bar{x}) + \bar{v} \nabla h(\bar{x}) = 0, \qquad (1)$$

$$\bar{u} g(\bar{x}) = 0, \qquad (2)$$

$$g(\bar{x}) \leqq 0, \qquad (3)$$

$$h(\bar{x}) = 0, \qquad (4)$$

$$\bar{u}_0, \bar{u}, \bar{v} \geqq 0, \qquad (5)$$

$$(\bar{u}_0, \bar{u}_P, \bar{v}_Q) \geqslant 0, \qquad (6)$$

Where

$$I = \{i : g_i(\bar{x}) = 0\},$$

$$P = \{i \in I : g_i \text{ is strictly pseudoconvex at } \bar{x}\} \text{ and}$$

Now

$$Q = \{j : h_j \text{ is strictly pseudoconvex at } \bar{x}\},$$

then $\bar{x}$ *solves Problem* (MP).

Proof. Let $J = \{i : g_i(\bar{x}) < 0\}$. Then

$$\bar{u} \geqq 0,\ g(\bar{x}) \leqq 0,\ \bar{u}g(\bar{x}) = 0 \Rightarrow \bar{u}_J = 0. \tag{7}$$

Suppose to the contrary that x is not an optimal solution for problem (MP). Then there exists x^0 ε X such that

$f(x^o) < f(x)$.

Using pseudoconvexity off at x, we get

Since x^o ε x

$$\nabla f(\bar{x})(x^\circ - \bar{x}) < 0.$$

$$g_I(x^\circ) \leqq 0 = g_I(\bar{x}), \tag{8}$$

$$h(x^\circ) = 0 = h(\bar{x}).$$

Strictly pseudoconvexity of g_P and h_Q at x implies

$$\nabla g_P(\bar{x})(x^\circ - \bar{x}) < 0, \tag{9}$$

and

$$\nabla h_Q(\bar{x})(x^\circ - \bar{x}) < 0, \tag{10}$$

and by quasiconvexity of $g_{P'}$ and $h_{Q'}$ at $\bar{x}$, we get

$$\nabla g_{P'}(\bar{x})(x^\circ - \bar{x}) \leqq 0, \tag{11}$$

$$\nabla h_{Q'}(\bar{x})(x^\circ - \bar{x}) \leqq 0, \tag{12}$$

where $P' = I - P$ and $Q' = \{1, 2, \ldots, k\} - Q$.

Now (5) to (12) give

$$[\bar{u}_0 \nabla f(\bar{x}) + \bar{u} \nabla g(\bar{x}) + \bar{v} \nabla h(\bar{x})](x^\circ - \bar{x}) < 0,$$

which contradicts (1). Hence x is an optimal solution for Problem (MP).

3. PARTICULAR CASES

1. IF P=Q= ϕ i.e. none of the components of g_I and h is strictly pseudoconxex, then (6) become u_0>0. In this case the above result is reduced to the Kuhn-Tucker type sufficient optimality theorem discussed by Bector and Gulati [2] and Singh [5].

2. If P= I and Q = {1,2,......,k} i.e., all the components of g_1 and h are strictly pseudoconvex at x ε X^0, then the above result is reduced to the Fritz John type sufficient optimality theorem discussed by Bector and Gulati [2] and Skarpness and Sposito [6]. Moreover, our proof avoids the use of an alternative theorem.

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